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TWO-DIMENSIONAL PROBLEM OF A UNIFORM FLOW OF A TWO-LAYER FLUID OF FINITE DEPTH PAST A CIRCULAR CYLINDER

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The linear steady problem of an irrotational uniform flow past a horizontal circular cylinder located in the upper or in the lower layer of a two-layer fluid is solved by the multipole-expansion method. The flow is perpendicular to the axis of the cylinder. The fluid is assumed to be inviscid and incompressible, and the flow in each layer is assumed to be potential. The upper layer can be bounded by a free surface or a solid lid, and the lower layer by a rigid horizontal bottom.

In recent years, the methods of calculating hydrodynamic loads acting on a submerged body of arbitrary shape which moves in a stratified fluid have intensely been developed [1–5]. In this connection, the problem of testing the numerical methods used is important and, therefore, it is useful to have a solution of the problem of a flow about a body of simple geometry that guarantees high accuracy of calculations. A circular cylinder is the most typical example of such a body in the two-dimensional case. The problem of a uniform flow of an unbounded two-layer fluid past a circular cylinder was solved explicitly in [6] using conformal mapping; however, this method not is applicable to a fluid of finite depth. In this case, it is convenient to use the multipole-expansion method (MEM) by means of which the problems of radiation and diffraction of surface waves by a submerged circular cylinder moving a in homogeneous deep fluid [7] were fully studied. The steady problem is a component of these problems. Wu [8] studied a uniform flow of an unbounded two-layer fluid about a circular cylinder. Linton and McIver [9] considered the diffraction of surface and internal waves by a circular cylinder in a two-layer fluid whose upper layer is of finite depth and whose lower layer is unbounded. The applications of the MEM to the solution of various wave problems were examined in [7–9]. The solutions obtained using the MEM are of great interest, because they allow one to analyze the effect of various parameters of the problem on the wave characteristics of the flow.

The method of modeling the boundaries by singularities is a numerical method that allows the problem under study for a cylinder of arbitrary cross section to be solved with any given accuracy [2]. However, for a circular cylinder the MEM requires minimum computing expenditures and can be extended to the three-dimensional problem of a flow about a sphere with a similar use of the point multipoles [10].

1. Formulation of the Problem. In an undisturbed state, the upper layer of a fluid of density ρ_1 and thickness H_1 occupies the region $|x| < \infty$, $0 < y < H_1$, and the lower layer of density $\rho_2 = \rho_1(1+\varepsilon)$ ($\varepsilon > 0$) and thickness H_2 occupies the region $|x| < \infty$, $-H_2 < y < 0$, where x and y are the horizontal and vertical coordinates. A uniform flow with velocity U runs against a body in the negative direction of the x axis. In each layer, the fluid flow is assumed to be potential, and the total velocity potential $\Phi_j(x,y)$ can be represented as

$$\Phi_j = -Ux + U\varphi_j,$$

where $\varphi_j(x,y)$ is the velocity potential corresponding to a uniform flow with unit velocity; the subscript $j = 1$

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and 2 is introduced for the upper and lower layers, respectively. In a fluid, we have

$$\Delta\varphi_1 = 0 \quad (0 < y < H_1), \qquad \Delta\varphi_2 = 0 \quad (-H_2 < y < 0). \tag{1.1}$$

According to the linear wave theory, the boundary conditions on the free surface and at the interface are satisfied on the horizontal planes corresponding to the undisturbed state of the fluid:

$$\frac{\partial^2\varphi_1}{\partial x^2} + \mu\frac{\partial\varphi_1}{\partial y} = 0 \qquad (y = H_1); \tag{1.2}$$

$$(1+\varepsilon)\frac{\partial^2\varphi_2}{\partial x^2} - \frac{\partial^2\varphi_1}{\partial x^2} + \varepsilon\mu\frac{\partial\varphi_1}{\partial y} = 0, \qquad \frac{\partial\varphi_1}{\partial y} = \frac{\partial\varphi_2}{\partial y} \qquad (y = 0). \tag{1.3}$$

The other boundary conditions are of the form

$$\frac{\partial\varphi_2}{\partial y} = 0 \qquad (y = -H_2) \tag{1.4}$$

at the bottom,

$$\frac{\partial\varphi_j}{\partial x} \to 0 \quad (x \to \infty), \quad \left|\frac{\partial\varphi_j}{\partial x}\right| < \infty \quad (x \to -\infty), \quad j = 1, 2 \tag{1.5}$$

in a far field, and

$$\frac{\partial\varphi_q}{\partial n} = n_x \qquad (x, y \in S) \tag{1.6}$$

on a circular contour S $(x^2 + [y + (-1)^q h]^2 = a^2)$.

In formulas (1.2)–(1.6), $\mu = g/U^2$, g is the acceleration of gravity, $\mathbf{n} = (n_x, n_y)$ is the internal normal to the body, a is the radius of the cylinder, h is the distance from the center of the cylinder to the interface ($h > a$), and $q = 1$ ($q = 2$) if the cylinder is in the upper (lower) layer.

If the upper layer is bounded by a rigid horizontal lid instead of the free surface, the boundary condition (1.2) becomes simpler:

$$\frac{\partial\varphi_1}{\partial y} = 0 \qquad (y = H_1).$$

The hydrodynamic forces $\mathbf{F} = (F_x, F_y)$ acting from the side of the fluid on the body in flow are determined by integrating the fluid pressure (without the hydrostatic term)

$$p = -\rho_q U^2\Big(\frac{\partial\varphi_q}{\partial x} - \frac{1}{2}|\nabla\varphi_q|^2\Big)$$

over the surface S, i.e.,

$$\mathbf{F} = \int_S p\,\mathbf{n}\,ds.$$

Introducing the polar coordinate system r, θ with origin in the center of the cylinder S

$$x = r\sin\theta, \quad y + (-1)^q h = r\cos\theta \qquad (q = 1, 2) \tag{1.7}$$

and taking into account that, for a circular cylinder

$$n_x = -\sin\theta, \qquad n_y = -\cos\theta, \tag{1.8}$$

similarly to [7] we obtain

$$(F_x, F_y) = \frac{\rho_q U^2}{2a}\int_0^{2\pi}\Big[\frac{\partial(\varphi_q - x)}{\partial\theta}\Big]^2(\sin\theta, \cos\theta)\,d\theta. \tag{1.9}$$

Below, we shall consider the solution of the formulated problem for a cylinder located in the upper and lower layers.

2. Cylinder in the Upper Layer. Using multipoles as the fundamental solutions of the Laplace equation, with allowance for the boundary condition (1.4) we write the solution of Eqs. (1.1) in the form

$$\varphi_1 = \sum_{m=1}^{\infty} a^m \Big[p_m\Big(\frac{\cos m\theta}{r^m} + f_m\Big) + q_m\Big(\frac{\sin m\theta}{r^m} + g_m\Big)\Big], \tag{2.1}$$

$$\varphi_2 = \sum_{m=1}^{\infty} a^m (p_m F_m + q_m G_m), \tag{2.2}$$

where

$$f_m = \frac{(-1)^m}{(m-1)!} \int_0^{\infty} k^{m-1} \cos kx \{A_1(k) \exp[k(y-h)] + B_1(k) \exp[k(h-y)]\}\, dk, \tag{2.3}$$

$$g_m = \frac{(-1)^{m+1}}{(m-1)!} \int_0^{\infty} k^{m-1} \sin kx \{A_2(k) \exp[k(y-h)] + B_2(k) \exp[k(h-y)]\}\, dk, \tag{2.4}$$

$$F_m = \frac{(-1)^m}{(m-1)!} \int_0^{\infty} k^{m-1} \cos kx C_1(k) \cosh k(y+H_2)\, dk, \tag{2.5}$$

$$G_m = \frac{(-1)^{m+1}}{(m-1)!} \int_0^{\infty} k^{m-1} \sin kx C_2(k) \cosh k(y+H_2)\, dk. \tag{2.6}$$

Using the known relations

$$\frac{\exp(im\theta)}{r^m} = \begin{cases} \dfrac{(-1)^m}{(m-1)!} \displaystyle\int_0^{\infty} k^{m-1} \exp[k(y-h-ix)]\, dk & (y < h), \\ \dfrac{1}{(m-1)!} \displaystyle\int_0^{\infty} k^{m-1} \exp[k(h-y+ix)]\, dk & (y > h) \end{cases} \tag{2.7}$$

and the boundary conditions (1.2) and (1.3), one can define the desired functions:

$$A_{1,2}(k) = \frac{(k+\mu)(1+t_1)}{2Z_1(k)} [T_1(k) \pm (-1)^m P_1(k) \exp(2kh)] \exp(-2kH_1); \tag{2.8}$$

$$B_{1,2}(k) = \frac{(1+t_1)T_1(k)}{2Z_1(k)} [(\mu - k) \exp(-2kh) \pm (-1)^m (k+\mu) \exp(-2kH_1)]; \tag{2.9}$$

$$C_j(k) = [(1+A_j) \exp(-kh) - B_j \exp(kh)]/\sinh kH_2 \quad (j = 1, 2). \tag{2.10}$$

The following notation is introduced:

$$Z_1(k) = (k^2 + \varepsilon\mu^2) t_1 t_2 + (1+\varepsilon) k [k - \mu(t_1 + t_2)], \tag{2.11}$$

$$(T_1, P_1) = (\varepsilon\mu \pm k) t_2 - k(1+\varepsilon), \quad t_1 = \tanh kH_1, \quad t_2 = \tanh kH_2.$$

The integrands in (2.3)–(2.6) can have simple poles which are the solutions of the equation

$$Z_1(k) = 0. \tag{2.12}$$

As is known (see, e.g., [11]), two critical velocities of the free flow U_1 and U_2 ($U_1 > U_2$) exist in a two-layer fluid of finite depth which is bounded by the free surface:

$$U_{1,2}^2 = \frac{g}{2}\Big[H \pm \sqrt{H^2 - \frac{4\varepsilon}{1+\varepsilon} H_1 H_2}\Big],$$

where $H = H_1 + H_2$. For $U > U_1$, the wave motions do not occur in the fluid; for $U_2 < U < U_1$, there is a wave with the maximum amplitude on the free surface (the surface wave), and, for $U < U_2$, in addition to it, an internal wave with the maximum amplitude at the interface appears.

Consequently, Eq. (2.12) has no real roots for $\mu < \mu_1$, has one root k_1 for $\mu_1 < \mu < \mu_2$, and two roots $k_{1,2}$ ($k_2 < k_1$) for $\mu > \mu_2$, where $\mu_{1,2} = g/U_{1,2}^2$. The root k_1 corresponds to the surface wave, and k_2 to the internal wave. In the integration in (2.3)–(2.6), the poles $k_{1,2}$ are gone around from below.

With allowance for the radiation condition in a far field (1.5), relations (2.3) and (2.4) take the form

$$f_m = \frac{(-1)^m}{(m-1)!}\Big\{\text{p.v.}\int_0^\infty k^{m-1}\cos kx[A_1\exp(k(y-h)) + B_1\exp(k(h-y))]\,dk$$

$$+\pi\sum_{l=1}^{2} k_l^{m-1}\sin k_l x[A_1^0(k_l)\exp(k_l(y-h)) + B_1^0(k_l)\exp(k_l(h-y))]\Big\}; \tag{2.13}$$

$$g_m = \frac{(-1)^{m+1}}{(m-1)!}\Big\{\text{p.v.}\int_0^\infty k^{m-1}\sin kx[A_2\exp(k(y-h)) + B_2\exp(k(h-y))]\,dk$$

$$-\pi\sum_{l=1}^{2} k_l^{m-1}\cos k_l x[A_2^0(k_l)\exp(k_l(y-h)) + B_2^0(k_l)\exp(k_l(h-y))]\Big\} \tag{2.14}$$

(the abbreviation p.v. means the integral as a principal value, and the superscript 0 refers to the residue of the corresponding function at the point $k = k_l$). In (2.13) and (2.14), the terms under the summation sign are taken into account only in the presence of the corresponding roots of Eq. (2.12).

This solution satisfies Eqs. (1.1) and all the boundary conditions of problem (1.2)–(1.5), except for the impermeability condition on the body. To make allowance for the boundary condition (1.6) on the surface of a body, the known relations

$$\exp[k(y-h+ix)] = \sum_{m=0}^{\infty}\frac{(kr)^m}{m!}\exp(im\theta),$$

$$\exp[k(h-y+ix)] = \sum_{m=0}^{\infty}(-1)^m\frac{(kr)^m}{m!}\exp(-im\theta)$$

are used. Equation (2.1) takes the form

$$\varphi_1 = \sum_{m=1}^{\infty} p_m a^m\frac{\cos m\theta}{r^m} + \sum_{m=0}^{\infty}\frac{r^m}{m!}\cos m\theta\sum_{n=1}^{\infty}\frac{(-1)^n a^n}{(n-1)!}\Big[p_n I_{n+m-1} + \pi q_n\sum_{l=1}^{2}k_l^{n+m-1}J_{n+m-1}(k_l)\Big]$$

$$+\sum_{m=1}^{\infty} q_m a^m\frac{\sin m\theta}{r^m} + \sum_{m=1}^{\infty}\frac{r^m}{m!}\sin m\theta\sum_{n=1}^{\infty}\frac{(-1)^n a^n}{(n-1)!}\Big[q_n M_{n+m-1} + \pi p_n\sum_{l=1}^{2}k_l^{n+m-1}L_{n+m-1}(k_l)\Big], \tag{2.15}$$

where

$$I_{n+m-1} = \text{p.v.}\int_0^\infty k^{n+m-1}[A_1 + (-1)^m B_1]\,dk; \tag{2.16}$$

$$M_{n+m-1} = \text{p.v.}\int_0^\infty k^{n+m-1}[(-1)^m B_2 - A_2]\,dk; \tag{2.17}$$

$$J_{n+m-1}(k_l) = A_2^0(k_l) + (-1)^m B_2^0(k_l); \tag{2.18}$$

$$L_{n+m-1}(k_l) = A_1^0(k_l) - (-1)^m B_1^0(k_l). \tag{2.19}$$

It is necessary to replace m by n in expressions (2.8) and (2.9) for $A_{1,2}$ and $B_{1,2}$.

Differentiating (2.15) with respect to r and taking into account (1.8), we obtain the following infinite system of linear equations for the determination of p_m and q_m:

$$p_m - \sum_{n=1}^{\infty}\frac{(-1)^n a^{m+n}}{m!(n-1)!}\Big[p_n I_{n+m-1} + \pi q_n\sum_{l=1}^{2}k_l^{n+m-1}J_{n+m-1}(k_l)\Big] = 0, \tag{2.20}$$

$$q_m - \sum_{n=1}^{\infty} \frac{(-1)^n a^{m+n}}{m!(n-1)!}\Big[q_n M_{n+m-1} + \pi p_n \sum_{l=1}^{2} k_l^{n+m-1} L_{n+m-1}(k_l)\Big] = -a\delta_{m1},$$

where δ_{m1} is the Kronecker symbol. In the numerical solution of this system, the reduction method is used, and only a finite number of terms necessary to achieve a given accuracy are taken into account.

After p_m and q_m are found, we calculate all the characteristics of the flow. Substituting relations (2.20) into (2.15), we obtain an expression for the potential on the hydrofoil:

$$\varphi_1\Big|_{r=a} = 2\sum_{m=1}^{\infty}(p_m \cos m\theta + q_m \sin m\theta) + a\sin\theta + \sum_{m=1}^{\infty} \frac{a^m}{(m-1)!}\Big[p_m I_{m-1} + \pi q_m \sum_{l=1}^{2} k_l^{m-1} J_{m-1}(k_l)\Big]. \quad (2.21)$$

With allowance for (1.9), the hydrodynamic forces are equal to

$$F_x = \frac{2\pi\rho_1 U^2}{a} \sum_{m=1}^{\infty} m(m+1)(p_m q_{m+1} - q_m p_{m+1}),$$
$$F_y = \frac{2\pi\rho_1 U^2}{a} \sum_{m=1}^{\infty} m(m+1)(p_m p_{m+1} + q_m q_{m+1}). \quad (2.22)$$

According to (2.1), the potential of the upper layer far from the hydrofoil has the form

$$\varphi_1 = \sum_{m=1}^{\infty} a^m (p_m f_m + q_m g_m) \qquad (|x| \to \infty);$$

$f_m = g_m = 0$ as $x \to \infty$, and

$$f_m = \frac{2\pi(-1)^m}{(m-1)!} \sum_{l=1}^{2} k_l^{m-1} \sin k_l x [A_1^0(k_l) \exp(k_l(y-h)) + B_1^0(k_l) \exp(k_l(h-y))],$$

$$g_m = \frac{2\pi(-1)^m}{(m-1)!} \sum_{l=1}^{2} k_l^{m-1} \cos k_l x [A_2^0(k_l) \exp(k_l(y-h)) + B_2^0(k_l) \exp(k_l(h-y))]$$

as $x \to -\infty$. With (2.10) taken into account, it is easy to determine the potential of the lower layer φ_2 in the far field. The wave motion exists only behind the body and, generally, represents a superposition of two waves: a surface wave and an internal wave with wave numbers k_1 and k_2, respectively. Knowing the potentials in the far field, one can determine the vertical displacements of the free surface $\eta_1(x)$ and the interfaces $\eta_2(x)$ far behind the body:

$$\eta_1(x) = \frac{U^2}{g} \frac{\partial\varphi_1}{\partial x}\Big|_{y=H_1}, \qquad \eta_2(x) = \frac{U^2}{\varepsilon g} \frac{\partial}{\partial x}[(1+\varepsilon)\varphi_2 - \varphi_1]_{y=0}.$$

We consider some particular cases of the solution of this problem. For $\varepsilon = 0$, we have a single-layer fluid of depth H. This case is the most studied, and calculations of various characteristics of the wave motion for a circular cylinder were given in [12–14]. In this flow, only a surface wave arises if the condition $U < \sqrt{gH}$ is satisfied.

If the lower layer of a two-layer fluid ($H_2 \to \infty$) has an infinite depth, in (2.11) we have $Z_1(k) = (\mu - k)[(\varepsilon\mu - k)t_1 - k(1+\varepsilon)]$. The surface wave is excited at any velocity of the incoming flow, and the pole $k_1 = \mu$ corresponds to this wave. The internal wave arises only for $U < \sqrt{\varepsilon g H_1/(1+\varepsilon)}$.

This solution is considerably simplified for an unbounded two-layer fluid ($H_1, H_2 \to \infty$). In this case, the internal wave exists at all flow velocities and relations (2.13) and (2.14) take the complex form

$$f_m + ig_m = \frac{(-1)^m\gamma}{(m-1)!}\Big[\text{p.v.} \int_0^{\infty} k^{m-1} \frac{k-\mu}{k-\nu} \exp(-k(y+h+ix))\,dk + i\pi\nu^{m-1}(\nu-\mu)\exp(-\nu(y+h+ix))\Big],$$

where $\nu = \gamma\mu$ and $\gamma = \varepsilon/(2+\varepsilon)$. We introduce the notation

$$S_m = p_m + iq_m. \quad (2.23)$$

System (2.20) takes the form

$$S_m - (-1)^m \gamma \sum_{n=1}^{\infty} (-1)^n \frac{a^{n+m} S_n}{m!(n-1)!} [I_{n+m-1} - i\pi\nu^{n+m-1}(\nu - \mu)\exp(-2\nu h)] = -ia\delta_{m1}.$$

Here

$$I_N = \text{p.v.} \int_0^{\infty} k^N \frac{k-\mu}{k-\nu} \exp(-2kh)\, dk$$

and is calculated by the recurrent formula

$$I_N = \Big(\frac{N}{2h} - \mu\Big) \frac{(N-1)!}{(2h)^N} + \nu I_{N-1}, \quad I_0 = \frac{1}{2h} - (\nu - \mu)\,\mathrm{Ei}\,(2\nu h)\, \exp(-2\nu h), \tag{2.24}$$

where $\mathrm{Ei}(z)$ is the integral exponential function (see, e.g. [15]). The hydrodynamic load calculations for this case were reported in [8].

It is of interest to study this problem in the limiting cases of $U \to 0$ ($\mu \to \infty$) and $U \to \infty$ ($\mu \to 0$). For $\mu \to \infty$, the gravitational forces dominate over the inertial ones, and the boundary conditions (1.2) and (1.3) take the form

$$\frac{\partial \varphi_1}{\partial y} = 0 \qquad (y = H_1, \quad y = 0).$$

Just as the interface, the free surface also becomes equivalent to a rigid lid, irrespective of the magnitude of the density jump. The problem is reduced to a flow about a circular hydrofoil located between horizontal rigid plates in a layer of thickness H_1. The integrals in (2.16) and (2.17) are reduced to the sum of the integrals of the form

$$V_N = \int_0^{\infty} k^N \Big(1 + \frac{1}{\tanh \beta k}\Big) \exp(-\alpha k)\, dk \qquad (N > 0, \quad \alpha > 0),$$

which can be calculated using the Riemann zeta-function ζ (formula (3.551(3)) from [16]):

$$V_N = \frac{N!}{2^N \beta^{N+1}} \zeta\Big(N + 1, \frac{\alpha}{2\beta}\Big).$$

It was noted in [3] that the lift force F_y undergoes a discontinuity for $\varepsilon = 0$ and as $\mu \to \infty$, because, assuming that $\varepsilon = 0$ and passing to the limit as $\mu \to \infty$, we obtain the flow about a body located between rigid plates in a layer of thickness H.

For a weightless fluid ($\mu \to 0$), the boundary condition (1.2) and the first condition in (1.3) at the interface take the form

$$\varphi_1 = 0 \quad (y = H_1), \qquad (1 + \varepsilon)\varphi_2 = \varphi_1 \quad (y = 0).$$

Here no wave motions are generated, and, hence, the wave resistance F_x is zero. For $\varepsilon = 0$, the discontinuity of the lift force is absent.

We also consider the case where the upper layer is limited from above by a rigid lid rather than the free surface. Here expressions (2.1)–(2.6) have the former form and instead of (2.8)–(2.11) we obtain

$$A_{1,2}(k) = \frac{1 + t_1}{2Z_2(k)} [T_1(k) \pm (-1)^m P_1(k) \exp(2kh)] \exp(-2kH_1),$$

$$B_{1,2}(k) = \frac{(1 + t_1)T_1(k)}{2Z_2(k)} [\exp(-2kh) \pm (-1)^m \exp(-2kH_1)],$$

where $Z_2(k) = \varepsilon\mu t_1 t_2 - -k[t_2 + (1 + \varepsilon)t_1]$. The only real root of the equation $Z_2(k) = 0$ exists only for $U < \sqrt{\varepsilon g H_1 H_2/[H_2 + (1 + \varepsilon)H_1]}$ and corresponds to the internal wave. Introducing these changes, we obtain the above solution.

3. Cylinder in the Lower Layer. If the cylinder is under the interface, the method of solution is very similar to that described above and we outline it briefly. We present the solution of Eqs. (1.1) in the form

$$\varphi_1 = \sum_{m=1}^{\infty} a^m (p_m F_m + q_m G_m), \quad \varphi_2 = \sum_{m=1}^{\infty} a^m \Big[p_m\Big(\frac{\cos m\theta}{r^m} + f_m\Big) + q_m\Big(\frac{\sin m\theta}{r^m} + g_m\Big)\Big],$$

where

$$F_m = \frac{1}{(m-1)!}\int_0^{\infty} k^{m-1} \cos kx [C_1(k)\cosh k(y-H_1) + D_1(k)\sinh k(y-H_1)]\,dk,$$

$$G_m = \frac{1}{(m-1)!}\int_0^{\infty} k^{m-1} \sin kx [C_2(k)\cosh k(y-H_1) + D_2(k)\sinh k(y-H_1)]\,dk,$$

$$f_m = \frac{1}{(m-1)!}\int_0^{\infty} k^{m-1} \cos kx \{A_1(k)\exp[k(y+h)] + B_1(k)\exp[-k(y+h)]\}\,dk,$$

$$g_m = \frac{1}{(m-1)!}\int_0^{\infty} k^{m-1} \sin kx \{A_2(k)\exp[k(y+h)] + B_2(k)\exp[-k(y+h)]\}\,dk.$$

Using (1.7), the analog of relations (2.7) for this case, and the boundary conditions (1.2)–(1.4), we obtain

$$A_{1,2}(k) = \frac{(k+\mu)(1+t_2)T_2(k)}{2Z_1(k)}[\exp(-2kh) \pm (-1)^m \exp(-2kH_2)],$$

$$B_{1,2}(k) = \frac{(1+t_2)}{2Z_1(k)}\exp(-2kH_2)[(k+\mu)T_2(k) \mp (-1)^m (k-\mu)P_2(k)\exp(2kh)], \tag{3.1}$$

$$C_j(k) = \frac{\mu[A_j\exp(kh) - (1+B_j)\exp(-kh)]}{k\cosh kH_1 - \mu\sinh kH_1}, \quad D_j = \frac{k}{\mu}C_j \qquad (j = 1, 2),$$

where $(T_2, P_2) = (\varepsilon\mu \pm k)t_1 - k(1+\varepsilon)$. The potential in the lower layer is

$$\varphi_2 = \sum_{m=1}^{\infty} p_m a^m \frac{\cos m\theta}{r^m} + \sum_{m=0}^{\infty} \frac{r^m}{m!}\cos m\theta \sum_{n=1}^{\infty} \frac{a^n}{(n-1)!}\Big[p_n I_{n+m-1} + \pi q_n \sum_{l=1}^{2} k_l^{n+m-1} J_{n+m-1}(k_l)\Big]$$

$$+ \sum_{m=1}^{\infty} q_m a^m \frac{\sin m\theta}{r^m} + \sum_{m=1}^{\infty} \frac{r^m}{m!}\sin m\theta \sum_{n=1}^{\infty} \frac{a^n}{(n-1)!}\Big[q_n M_{n+m-1} + \pi p_n \sum_{l=1}^{2} k_l^{n+m-1} L_{n+m-1}(k_l)\Big]. \tag{3.2}$$

The expressions for I_N and L_N coincide in form with those given in (2.16) and (2.19), and the expressions for M_N and J_N differ from (2.17) and (2.18) by the sign.

The system for determining p_m and q_m has the form

$$p_m - - \sum_{n=1}^{\infty} \frac{a^{m+n}}{m!(n-1)!}\Big[p_n I_{n+m-1} + \pi q_n \sum_{l=1}^{2} k_l^{n+m-1} J_{n+m-1}(k_l)\Big] = 0,$$

$$q_m - - \sum_{n=1}^{\infty} \frac{a^{m+n}}{m!(n-1)!}\Big[q_n M_{n+m-1} + \pi p_n \sum_{l=1}^{2} k_l^{n+m-1} L_{n+m-1}(k_l)\Big] = -a\delta_{m1}. \tag{3.3}$$

For the potential φ_2 on the cylinder surface, expression (2.21) is true. After replacing ρ_1 by ρ_2, the relations for the hydrodynamic forces coincide with (2.22). For a boundless two-layer fluid ($H_1, H_2 \to \infty$), we have

$$f_m + ig_m = -\frac{\gamma}{(m-1)!}\Big[\text{p.v.}\int_0^{\infty} k^{m-1}\frac{k+\mu}{k-\nu}\exp(k(y-h+ix))\,dk - i\pi\nu^{m-1}(\nu+\mu)\exp(\nu(y-h+ix))\Big].$$

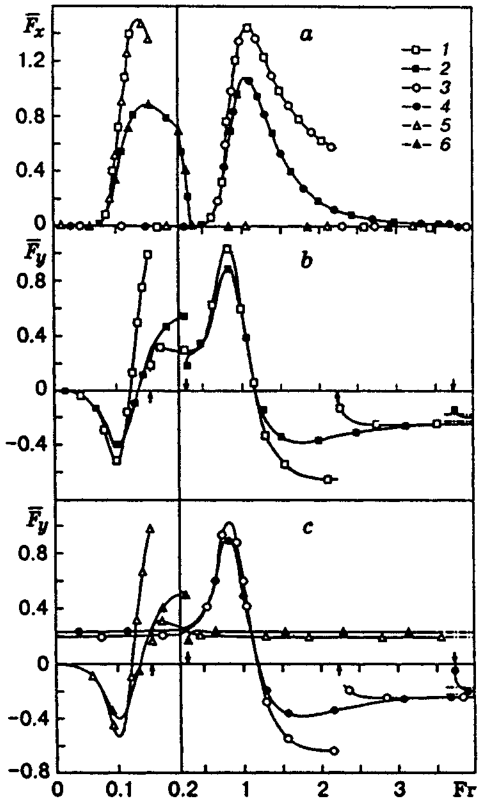

Fig. 1

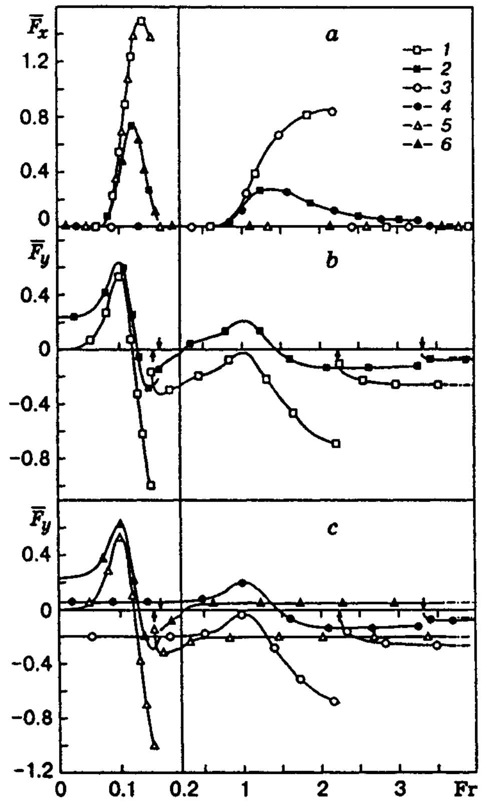

Fig. 2

The system for determining S_m (2.23) has the form

$$S_m + \gamma \sum_{n=1}^{\infty} \frac{a^{n+m} S_n}{m!(n-1)!} [I_{n+m-1} + i\pi\nu^{n+m-1}(\nu+\mu)\exp(-2\nu h)] = -ia\delta_{m1},$$

where

$$I_N = \text{p.v.} \int_0^{\infty} k^N \frac{k+\mu}{k-\nu} \exp(-2kh)\, dk.$$

The values of I_N can be found by recurrent formulas similar to (2.24). The hydrodynamic-load calculations for an unbounded two-layer fluid were given in [6, 8].

With the free surface replaced by a rigid lid, instead of (3.1) we obtain

$$A_{1,2}(k) = \frac{T_3(1+t_2)}{2Z_2(k)} [\exp(-2kh) \pm (-1)^m \exp(-2kH_2)],$$

$$B_{1,2}(k) = \frac{(1+t_2)}{2Z_2(k)} [T_3 \pm (-1)^m \exp(2kh)] \exp(-2kH_2),$$

$$C_j = [(1+B_j)\exp(-kh) - A_j \exp(kh)]/\sinh kH_1, \quad D_j = 0 \quad (j = 1, 2),$$

where $(T_3, P_3) = [\mu\varepsilon \pm k(1+\varepsilon)]t_1 - -k$. With allowance for these changes, expressions (3.2) and (3.3) are reproduced in this case.

4. Numerical Calculations. The numerical results for the particular cases of these problems coincide with known results [6–8, 12–14].

Calculations of the hydrodynamic loads versus the Froude number $\mathrm{Fr} = U/\sqrt{ga}$ are shown in Fig. 1 for a cylinder located in the upper layer and in Fig. 2 for a cylinder located in the lower layer. We introduce the notation $(\bar{F}_x, \bar{F}_y) = (-F_x, F_y)/\rho_q a U^2$ ($q = 1, 2$) and various scales on the abscissa in the intervals $[0; 0.2]$ and $[0, 2.4]$.

Curves 1 and 2 in Figs. 1 and 2 refer to the calculation results for the case where the upper layer is bounded by a free surface for $\varepsilon = 0.03$, curves 3 and 4 to similar results for $\varepsilon = 0$ (the one-layer fluid of depth H), and curves 5 and 6 to the calculation results for a two-layer fluid under a lid ($\varepsilon = 0.03$). The calculation results in Fig. 1 were obtained for constant values of the thickness of the upper layer and the depth of submergence of the cylinder: $H_1/a = 4$ and $h/a = 2$. The odd curves correspond to $H_2/a = 1$, and the even curves to $H_2/a = 10$. The arrows indicate the values of the critical Froude numbers; the upward arrows refer to $H_2/a = 1$, and the downward ones to $H_2/a = 10$. Figure 2 shows calculation results for the cylinder in the lower layer for $H_1/a = 1$ and $h/a = 2$. The odd curves correspond to $H_2/a = 4$, and the even curves to $H_2/a = 10$. In Fig. 2, the upward and downward arrows refer to the critical Froude numbers for $H_2/a = 4$ and $H_2/a = 10$, respectively. Figures 1b and 2b indicate the critical Froude numbers for a two-layer fluid bounded by a free surface, and Figs. 1c and 2c for a two-layer fluid under a lid and a one-layer fluid with a free surface. The horizontal dot-and-dashed curves in Figs. 1 and 2 show the limiting values of the lift force of the cylinder in a weightless fluid ($\mathrm{Fr} \to \infty$).

Comparison with the results of [6] showed that, to attain a calculational accuracy of up to 10^{-4}, it suffices to take into account only eight terms in the solution of the linear systems (2.20) and (3.3).

The special feature of the components of the hydrodynamic load in a fluid of finite depth is the appearance of discontinuities upon passage through the critical velocities. It shows up most strikingly for comparable dimensions of the cylinder diameter and the entire depth of the fluid. As the layer thickness increases, the discontinuities in the values of the hydrodynamic forces decrease and disappear at an infinite depth (see, e.g., [1]). The discontinuities in the values of the hydrodynamic forces upon passage through the critical velocity are the shortcoming of the linear approximation used, and it was suggested in [13] to take into account the nonlinear effects in the neighborhood of these velocities by a longwave approximation.

The wave resistance is markedly increased as the full depth of the fluid decreases (Figs. 1a and 2a), which is attributed to the increase in the horizontal velocity in the neighborhood of the body owing to the effect of flow encumbering and the resulting increase in the wave amplitudes (for details, see [14]).

It follows from the analysis of Figs. 1 and 2 that the stratification effect is insignificant at large flow velocities when only the surface wave is generated, and the "rigid lid" approximation satisfactorily describes the hydrodynamic loading at small flow velocities when the internal wave is generated most intensely.

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